Non-Linearly Extended Self-Dual Relations From The Nambu-Bracket Description Of M5-Brane In A Constant C-Field Background

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Abstract

The derivation of the self-dual relations for the two-form gauge field in the Nambu-bracket description of M5-brane in a constant C-field background initiated in Ref.[1] is completed by including contributions from all the fields in the M5-brane action. The result is used to examine Seiberg-Witten map of the BPS conditions for the string solitons, up to the first order in the expansion by the parameter g which characterizes the strength of the interactions through the Nambu-bracket.

1 Introduction

M-theory has been providing us important insights into the non-perturbative aspects of string theory. However, its microscopic definition is still lacking. M2-branes and M5-branes are fundamental building blocks of M-theory. Considering the current status, any new information on their properties could be important for finding more fundamental formulation of M-theory.

Recent few years have seen a rapid progress in the description of M-theory branes: A model for multiple M-theory membranes with a symmetry based on Lie 3-algebra was proposed in Refs.[2, 3, 4]. Starting from the BLG model, an action for M5-brane was constructed in Ho-Matsuo [5] and Ho-Imamura-Matsuo-Shiba [6] by taking the Lie 3-algebra which is defined through the Nambu-bracket [7].[1] This is in parallel with the construction of a Dp-brane in a constant B-field background from infinitely many D$(p-2)$-branes [9, 10, 11, 12]. The low energy effective action on the Dp-brane is given by Yang-Mills theory on non-commutative space. In fact, soon after the discovery of the non-commutative description of the D-brane worldvolume theory, the uplift to M-theory, namely M5-brane in a constant C-field background, was also investigated by several groups.[2] What was missing at the time was the appropriate uplift of the non-commutative description to that for the worldvolume theory of M5-brane, which now we have a candidate.

Interestingly, in the case of a D-brane in a constant B-field background, there is also an S-matrix equivalent description on a space with ordinary commutative coordinates. The map between the non-commutative description and the ordinary description is called Seiberg-Witten map [20].[3] With this historical background in mind, Ho et al. conjectured that the Nambu-bracket description of M5-brane is related to the previously found ordinary description of M5-brane [22, 23, 24, 25, 26, 27, 28, 29] in a constant C-field background via a straightforward generalization of the Seiberg-Witten map. The first non-trivial check of this conjecture was made in Ref.[30] for the BPS string-like soliton configurations on the M5-brane [31, 32, 33], which describe M2-branes ending on the M5-brane.

A peculiar feature of the M5-brane action of Refs.[5, 6] was that some components of the two-form gauge field were absent. In Ref.[1], it was demonstrated how the missing components of the two-form gauge field as well as self-dual relations for the field strength of the two-form gauge field can be obtained from this M5-brane action. The self-dual relations for the two-form gauge field are characteristic feature of M5-brane, and how to describe the self-dual two-form gauge field [34, 35, 36, 37, 38] was a central issue in the previous constructions of the M5-brane action [24, 25, 26]. It will be also important to clarify how the self-dual relations are maintained in the Nambu-bracket description of M5-brane.

In this paper, the derivation of the self-dual relations initiated in Ref.[1] is completed by including contributions from all the fields in the Nambu-bracket description of M5-brane. The

[1] Also see Ref.[8] for a similar construction of M5-brane action in a different background.

[2] A partial list includes [13, 14, 15, 16, 17, 18]. Also see Ref.[19] for a study after the recent developments of M-theory brane models.

[3] A similar map has appeared in the study of lowest Landau level fermions [21].

necessity of the inclusion of the scalar fields to the self-dual relations can be understood considering the fact that the scalar fields are related to the embedding coordinate fields in the ordinary description of M5-brane via the Seiberg-Witten map: In the ordinary description of M5-brane, the self-dual relations involve the embedding coordinate fields through the induced metric on the M5-brane. The obtained self-dual relations in the Nambu-bracket description of M5-brane are then used to examine the conjectured equivalence between the Nambu-bracket description and the ordinary description of M5-brane via the Seiberg-Witten map, in the case of the BPS conditions for the string solitons. Since the string solitons involve non-trivial configurations of a scalar field, the inclusion of the contribution of the scalar fields in the self-dual relations is essential.

2 The Nambu-bracket description of M5-brane in a constant C-field background

In this section I review the Nambu-bracket description of M5-brane in a constant C-field background constructed in Ref.[6] and fix my notation. (Also see Ref.[39] for a concise review).

The action given in Ref.[6] describes an M5-brane in the eleven-dimensional Minkowski space whose worldvolume extends in one time and five spacial directions. The direction along the worldvolume are parametrized by coordinates x^a $(a = 0, 1, 2)$ and $y^{\dot{a}}$ $(\dot{a} = 3, 4, 5)$. The metric on the worldvolume is specified by the components $\eta_{ab} = \mathrm{diag}(-++)$, $\delta_{\dot{a}\dot{b}}$, and other components are zero. There is a constant C-field background, with only C_{012} and C_{345} components are non-zero. Although both C_{012} and C_{345} components are turned on, the treatments of 012 directions and 345 directions are quite asymmetric, which might be one of the reasons why this action was not discovered until recently. The field content of the Nambu-bracket description of M5-brane is as follows: The scalars X^I $(I = 6, \cdots, 10)$ describe embedding coordinates transverse to the M5-brane worldvolume. The six-dimensional chiral fermions can be conveniently parametrized by a single eleven-dimensional Majorana spinor Ψ satisfying

$$\Gamma\Psi = -\Psi, \tag{2.1}$$

where Γ is given by

$$\Gamma = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5. \tag{2.2}$$

The Γ-matrices are those for the eleven-dimensional space-time. The salient feature of the M5-brane worldvolume theory is the self-duality of the two-form gauge field A, which is the focus of this paper. The components of the self-dual two-form gauge field A should be given by A_{ab}, $A_{a\dot{b}}$, $A_{\dot{a}\dot{b}}$, but the components A_{ab} do not appear in the action. They will appear from the equations of motion, as will be described in the next section.

The M5-brane action is given as follows:

$$S = \frac{T_{M5}}{g^2}\left(S_B + S_{CS} + S_F\right), \tag{2.3}$$

where

$$
\begin{aligned}
S_B &= \int d^3x d^3y \Bigg[-\frac{1}{2}\mathcal{D}_a X^I \mathcal{D}^a X^I - \frac{1}{2}\mathcal{D}_{\dot a} X^I \mathcal{D}^{\dot a} X^I \\
&\quad -\frac{1}{4}\mathcal{H}_{a\dot b\dot c}\mathcal{H}^{a\dot b\dot c} - \frac{1}{12}\mathcal{H}_{\dot a\dot b\dot c}\mathcal{H}^{\dot a\dot b\dot c} - \frac{g^4}{4}\{X^{\dot a}, X^I, X^J\}\{X^{\dot a}, X^I, X^J\} \\
&\quad -\frac{g^4}{12}\{X^I, X^J, X^K\}\{X^I, X^J, X^K\} \Bigg], \qquad (2.4) \\
S_{CS} &= -\int d^3x d^3y \left[\frac{1}{2}\epsilon^{abc} B_a{}^{\dot a}\partial_b A_{c\dot a} + g \det B_a{}^{\dot a} \right], \qquad (2.5) \\
S_F &= \int d^3x d^3y \Bigg[\frac{i}{2}\bar\Psi\Gamma^a\mathcal{D}_a\Psi + \frac{i}{2}\bar\Psi\Gamma^{\dot a}\mathcal{D}_{\dot a}\Psi \\
&\quad + \frac{ig^2}{2}\bar\Psi\Gamma_{\dot a I}\{X^{\dot a}, X^I, \Psi\} - \frac{ig^2}{4}\bar\Psi\Gamma_{IJ}\Gamma_{345}\{X^I, X^J, \Psi\} \Bigg]. \qquad (2.6)
\end{aligned}
$$

$\{*,*,*\}$ is the Nambu-bracket on $\mathbb{R}^3$:

$$
\{f, g, h\} = \epsilon^{\dot a\dot b\dot c}\frac{\partial}{\partial y^{\dot a}}f\,\frac{\partial}{\partial y^{\dot b}}g\,\frac{\partial}{\partial y^{\dot c}}h, \qquad (2.7)
$$

with $\epsilon^{345} = 1$. The covariant derivatives in the action are given as

$$
\begin{aligned}
\mathcal{D}_a\varphi &= (\partial_a - gB_a{}^{\dot a}\partial_{\dot a})\varphi, \quad (\varphi = X^I, \Psi), \qquad (2.8) \\
\mathcal{D}_{\dot a}\varphi &= \frac{g^2}{2}\epsilon_{\dot a\dot b\dot c}\{X^{\dot b}, X^{\dot c}, \varphi\}, \qquad (2.9)
\end{aligned}
$$

where

$$
X^{\dot a} = \frac{y^{\dot a}}{g} + A^{\dot a}, \quad A^{\dot a} = \frac{1}{2}\epsilon^{\dot a\dot b\dot c}A_{\dot b\dot c}\,, \qquad (2.10)
$$

and

$$
B_a{}^{\dot a} = \epsilon^{\dot a\dot b\dot c}\partial_{\dot b}A_{a\dot c}\,. \qquad (2.11)
$$

It follows that $\partial_{\dot c}B_a{}^{\dot c} = 0$. When one derives the M5-brane action from the BLG model, the components of the two-form gauge field $A_{a\dot b}$ arise from the gauge field in the BLG model [6].

The gauge transformation laws are given as

$$
\begin{aligned}
\delta_\Lambda\varphi &= g\kappa^{\dot c}\partial_{\dot c}\varphi, \quad (\varphi = X^I, \Psi), \qquad (2.12) \\
\delta_\Lambda A_{\dot a\dot b} &= \partial_{\dot a}\Lambda_{\dot b} - \partial_{\dot b}\Lambda_{\dot a} + g\kappa^{\dot c}\partial_{\dot c}A_{\dot a\dot b}\,, \qquad (2.13) \\
\delta_\Lambda A_{a\dot b} &= \partial_a\Lambda_{\dot b} - \partial_{\dot b}\Lambda_a + g\kappa^{\dot c}\partial_{\dot c}A_{a\dot b} + g(\partial_{\dot b}\kappa^{\dot c})A_{a\dot c}\,, \qquad (2.14)
\end{aligned}
$$

where

$$
\kappa^{\dot a} = \epsilon^{\dot a\dot b\dot c}\partial_{\dot b}\Lambda_{\dot c}\,. \qquad (2.15)
$$

It follows that $\partial_{\dot{c}}\kappa^{\dot{c}} = 0$. Thus, the gauge transformation by the parameter κ generates volume-preserving diffeomorphisms, and $B_a{}^{\dot{a}}$ is the gauge field for the volume-preserving diffeomorphisms:

$$\delta_\Lambda y^{\dot{a}} = g\kappa^{\dot{a}}. \tag{2.16}$$

The transformation law of $B_a{}^{\dot{a}}$ under the volume-preserving diffeomorphisms follows from eq.(2.14):

$$\delta_\Lambda B_a{}^{\dot{a}} = \partial_a\kappa^{\dot{a}} + g\kappa^{\dot{b}}\partial_{\dot{b}}B_a{}^{\dot{a}} - gB_a{}^{\dot{b}}\partial_{\dot{b}}\kappa^{\dot{a}}\,. \tag{2.17}$$

From eq.(2.17) it follows that the covariant derivatives (2.8) and (2.9) transform as scalars under the volume-preserving diffeomorphisms (2.16) [6, 1]. This allows one to construct gauge field strengths which transform as scalars under the volume-preserving diffeomorphisms.

It is worth mentioning a subtle point here, which was nicely explained in Ref.[1]: Although the fields $X^{\dot{a}}$ carry index $\dot{a}$, they transform as scalars under the volume-preserving diffeomorphisms (2.16). Indeed, in the derivation of the M5-brane action from the BLG model in Ref.[6], initially the target space indices of the scalar fields X in the BLG model (with the Nambu-bracket as the Lie 3-algebra) have nothing to do with the indices of the coordinates $y^{\dot{a}}$ on the Nambu-Poisson manifold ($\mathbb{R}^3$ in current case). What relates these different types of indices is the background values of the scalar fields X:

$$X^{\dot{a}}_{bg} = \frac{y^{\dot{a}}}{g}, \quad \dot{a} = 3, 4, 5. \tag{2.18}$$

Since the identification of the indices in eq.(2.18) was made in a particular choice of coordinates, it would appear different if one makes a volume-preserving reparametrization of the coordinates. However, one can keep the identification (2.18) intact instead, and absorb the induced change into the transformation of the "fluctuation" part $A^{\dot{a}}$ of the field $X^{\dot{a}}$ (2.10). This was how the gauge transformation law (2.13) arose from the scalar fields $X^{\dot{a}}$.[4]

The field strengths for the two-form gauge field which are made of the components $A_{a\dot{b}}$ and

[4]It may be useful to make a comment on a related but different topic for clarification. One may try to compare the Nambu-bracket action of M5-brane with the DBI-type action of M5-brane which has *the worldvolume reparametrization invariance*, as discussed in the Discussions section (note that the BLG model with the Nambu-bracket only has the invariance under the volume-preserving diffeomorphisms in the dotted directions). In this case, $y^{\dot{a}}$ may be interpreted as (a part of) the worldvolume coordinates of the M5-brane. From the results in D-branes in a constant B-field background [40, 41, 42, 43, 44], it is expected that by choosing the so-called static gauge $X^{\dot{a}} = y^{\dot{a}}/g$ for the worldvolume reparametrization of the DBI-type M5-brane action, one obtains the "commutative" description of M5-brane. In this description, the fluctuation of the Nambu-Poisson structure is parametrized by the ordinary gauge field in the DBI-type action. On the other hand, one can choose the gauge where the Nambu-Poisson structure is fixed. The fluctuation part $A^{\dot{a}}$ of the scalar field cannot be eliminated in this gauge: $X^{\dot{a}} = y^{\dot{a}}/g + A^{\dot{a}}$. The volume-preserving part of the worldvolume reparametrization remains as a residual symmetry, and one obtains the Nambu-bracket description of M5-brane. Please see the above mentioned papers for more detail in the case of D-branes in a constant B-field background. In the case of M5-brane, the detail has not been worked out at this moment.

$A_{\dot{a}\dot{b}}$ are given by

$$\mathcal{H}_{a\dot{b}\dot{c}} = \epsilon_{\dot{b}\dot{c}\dot{d}}\mathcal{D}_a X^{\dot{d}} = F_{a\dot{b}\dot{c}} - gB_a{}^{\dot{d}}\partial_{\dot{d}}A_{\dot{b}\dot{c}}\,, \tag{2.19}$$

$$\begin{aligned}\mathcal{H}_{\dot{a}\dot{b}\dot{c}} &= g^2\epsilon_{\dot{a}\dot{b}\dot{c}}\left(\{X^3,X^4,X^5\} - \frac{1}{g}\right)\\ &= F_{\dot{a}\dot{b}\dot{c}} + g\epsilon_{\dot{a}\dot{b}\dot{c}}\left((\partial_{\dot{f}}A^{\dot{f}})\partial_{\dot{g}}A^{\dot{g}} - (\partial_{\dot{f}}A^{\dot{g}})\partial_{\dot{g}}A^{\dot{f}}\right) + g^2\epsilon_{\dot{a}\dot{b}\dot{c}}\{A^3,A^4,A^5\},\end{aligned} \tag{2.20}$$

where $F_{a\dot{b}\dot{c}}$ and $F_{\dot{a}\dot{b}\dot{c}}$ are components of the linear part of the field strength:

$$F_{a\dot{b}\dot{c}} = \partial_a A_{\dot{b}\dot{c}} - \partial_{\dot{b}}A_{a\dot{c}} + \partial_{\dot{c}}A_{a\dot{b}}\,, \tag{2.21}$$

$$F_{\dot{a}\dot{b}\dot{c}} = \partial_{\dot{a}} A_{\dot{b}\dot{c}} + \partial_{\dot{b}}A_{\dot{c}\dot{a}} + \partial_{\dot{c}}A_{\dot{a}\dot{b}}\,. \tag{2.22}$$

As mentioned above, the field stengths (2.19) and (2.20) transform as scalars under the volume-preserving diffeomorphisms (2.16).

It is convenient to define a matrix $M_{\dot{a}}{}^{\dot{b}}$ following Ref.[1]:

$$M_{\dot{a}}{}^{\dot{b}} = g\partial_{\dot{a}}X^{\dot{b}}. \tag{2.23}$$

The matrix $M_{\dot{a}}{}^{\dot{b}}$ transforms as a vector with respect to the lower index $\dot{a}$:

$$\delta_\Lambda M_{\dot{a}}{}^{\dot{b}} = g\kappa^{\dot{c}}\partial_{\dot{c}}M_{\dot{a}}{}^{\dot{b}} + g(\partial_{\dot{a}}\kappa^{\dot{c}})M_{\dot{c}}{}^{\dot{b}}. \tag{2.24}$$

Because of this property the matrix $M_{\dot{a}}{}^{\dot{b}}$ and its inverse can be used as a "bridge" which converts scalar quantities to vector quantities, and vice versa. In particular, the following identity holds:

$$\mathcal{D}_{\dot{a}}\varphi = \det M M^{-1}{}_{\dot{a}}{}^{\dot{b}}\partial_{\dot{b}}\varphi. \tag{2.25}$$

The equations of motion of X^I and Ψ following from the action (2.3) are

$$\begin{aligned}0 &= \mathcal{D}_a\mathcal{D}^a X^I + \mathcal{D}_{\dot{a}}\mathcal{D}^{\dot{a}}X^I\\ &\quad + g^4\{X^{\dot{a}},X^J,\{X^{\dot{a}},X^J,X^I\}\} + \frac{g^4}{2}\{X^J,X^K,\{X^J,X^K,X^I\}\}\\ &\quad + \frac{ig^2}{2}\{\bar{\Psi}\Gamma_{\dot{a}I},X^{\dot{a}},\Psi\} + \frac{ig^2}{2}\{\bar{\Psi}\Gamma_{IJ}\Gamma_{345},X^J,\Psi\},\end{aligned} \tag{2.26}$$

$$0 = \Gamma^a\mathcal{D}_a\Psi + \Gamma^{\dot{a}}\mathcal{D}_{\dot{a}}\Psi + g^2\Gamma_{\dot{a}I}\{X^{\dot{a}},X^I,\Psi\} - \frac{g^2}{2}\Gamma_{IJ}\Gamma_{345}\{X^I,X^J,\Psi\}. \tag{2.27}$$

The equations of motion of gauge fields $A_{a\dot{b}}$ and $A_{\dot{a}\dot{b}}$ and the Bianchi identity can be written as

$$\mathcal{D}_a\mathcal{H}^{a\dot{b}\dot{c}} + \mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}\dot{b}\dot{c}} = gJ^{\dot{b}\dot{c}}, \tag{2.28}$$

$$\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot{c}} + \mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}b\dot{c}} = gJ^{b\dot{c}}, \tag{2.29}$$

$$\mathcal{D}_a\tilde{\mathcal{H}}^{abc} + \mathcal{D}_{\dot{a}}\tilde{\mathcal{H}}^{\dot{a}bc} = 0, \tag{2.30}$$

where $\mathcal{H}^{\dot{a}b\dot{c}} = -\mathcal{H}^{b\dot{a}\dot{c}}$ and

$$J^{\dot{a}\dot{b}} = J_B^{\dot{a}\dot{b}} + J_F^{\dot{a}\dot{b}}, \quad J^{a\dot{b}} = J_B^{a\dot{b}} + J_F^{a\dot{b}}, \tag{2.31}$$

$$J_B^{\dot{a}\dot{b}} = g\left(\{X^I, \mathcal{D}^{\dot{a}}, X^{\dot{b}}\} - (\dot{a} \leftrightarrow \dot{b})\right) - \frac{g^3}{2}\epsilon^{\dot{a}\dot{b}\dot{c}}\{X^I, X^J, \{X^I, X^J, X^{\dot{c}}\}\}, \tag{2.32}$$

$$J_F^{\dot{a}\dot{b}} = \frac{ig}{2}\left(\{\bar{\Psi}\Gamma^{\dot{a}}, X^{\dot{b}}, \Psi\} - (\dot{a} \leftrightarrow \dot{b})\right) + \frac{ig^2}{2}\epsilon^{\dot{a}\dot{b}\dot{c}}\{\bar{\Psi}\Gamma_{\dot{c}I}, X^I, \Psi\}, \tag{2.33}$$

$$J_B^{a\dot{b}} = g\{X^I, \mathcal{D}^a X^I, X^{\dot{b}}\}, \tag{2.34}$$

$$J_F^{a\dot{b}} = \frac{ig}{2}\{\bar{\Psi}\Gamma^a, \Psi, X^{\dot{b}}\}. \tag{2.35}$$

The Hodge dual on the six-dimensional M5-brane worldvolume is defined through the totally anti-symmetric tensor $\epsilon_{\mu\nu\rho\lambda\sigma\delta}$ $(\mu, \nu = 0, 1, \cdots, 5)$:

$$\epsilon_{abc\dot{a}\dot{b}\dot{c}} = -\epsilon_{\dot{a}\dot{b}\dot{c}abc} = \epsilon_{a\dot{b}\dot{c}bc\dot{a}} = \epsilon_{abc}\epsilon_{\dot{a}\dot{b}\dot{c}}\,, \tag{2.36}$$

with $\epsilon_{012} = -\epsilon^{012} = -1$. A three-form $\mathcal{H}$ is said to be (linearly) self-dual when it satisfies

$$\tilde{\mathcal{H}}_{\mu\nu\rho} = \mathcal{H}_{\mu\nu\rho}\,, \tag{2.37}$$

where

$$\tilde{\mathcal{H}}_{\mu\nu\rho} = \frac{1}{6}\epsilon_{\mu\nu\rho\lambda\sigma\delta}\mathcal{H}^{\lambda\sigma\delta}. \tag{2.38}$$

3 Non-linearly extended self-dual relations from the Nambu-bracket description of M5-brane

In Ref.[1], it was shown how the missing components of the two-form gauge field as well as (the non-linear extension of) the self-dual relations appear from the Nambu-bracket description of M5-brane action. Their analysis was restricted to the two-form gauge field part of the action. In the following, I will extend their results by including contributions from all the fields in the M5-brane action. The readers are recommended to go through the relevant part of Ref.[1].

Below I explain calculations involving only bosonic fields in some detail. Calculations involving fermions are bit lengthy but similar, so I will just quote the final result in the appendix A.2.

Following Ref.[1], I start from multiplying $M^{-1\dot{d}}_{\dot{c}}$ to eq.(2.29), where the matrix $M_{\dot{a}}{}^{\dot{b}}$ was defined in eq.(2.23):

$$M^{-1\dot{d}}_{\dot{c}}\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot{c}} + M^{-1\dot{d}}_{\dot{c}}\mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}b\dot{c}} = gM^{-1\dot{d}}_{\dot{c}}J_B^{b\dot{c}}. \tag{3.1}$$

Here, as mentioned above, I consider the case involving only bosonic fields, so only $J_B^{b\dot{c}}$ in $J^{b\dot{c}}$ (2.31) is taken into account. In Ref.[1], it was shown that the left hand side of eq.(3.1) can be written as a total derivative:

$$\begin{aligned} & M^{-1\dot{d}}_{\dot{c}}\mathcal{D}_a\tilde{\mathcal{H}}^{ab\dot{c}} + M^{-1\dot{d}}_{\dot{c}}\mathcal{D}_{\dot{a}}\mathcal{H}^{\dot{a}b\dot{c}} \\ = {} & \frac{1}{2}\epsilon^{\dot{a}\dot{b}\dot{d}}\partial_{\dot{a}}(M_{\dot{b}}{}^{\dot{f}}\epsilon_{\dot{f}\dot{g}\dot{k}}\mathcal{H}^{b\dot{g}\dot{k}}) - \epsilon^{bac}\epsilon^{\dot{a}\dot{b}\dot{d}}\partial_{\dot{a}}\left(\partial_a A_{c\dot{b}} + \frac{g}{2}\epsilon_{\dot{b}\dot{f}\dot{g}}B_a{}^{\dot{f}}B_c^{\dot{g}}\right). \end{aligned} \tag{3.2}$$

On the other hand, the right hand side of eq.(3.1) can also be written as a total derivative:

$$\begin{aligned} gM_{\dot{c}}^{-1\dot{d}}J_B^{b\dot{c}} &= g^2M_{\dot{c}}^{-1\dot{d}}\epsilon^{\dot{e}\dot{f}\dot{g}}(\partial_{\dot{e}}X^I)(\partial_{\dot{f}}D^bX^I)\partial_{\dot{g}}X^{\dot{c}} \\ &= g\epsilon^{\dot{e}\dot{f}\dot{d}}(\partial_{\dot{e}}X^I)\partial_{\dot{f}}D^bX^I = g\partial_{\dot{f}}(\epsilon^{\dot{e}\dot{f}\dot{d}}(\partial_{\dot{e}}X^I)D^bX^I) \\ &= -\epsilon^{bac}\epsilon^{\dot{a}\dot{b}\dot{d}}\partial_{\dot{a}}\left(\frac{g}{2}\epsilon_{acd}(\partial_{\dot{b}}X^I)D^dX^I\right). \end{aligned} \tag{3.3}$$

Notice the convention for the anti-symmetric tensor ϵ^{abc} (see appendix A.1). Eq.(3.2) and eq.(3.3) are total derivatives. By the Poincaré lemma one obtains

$$\mathcal{H}^{ab\dot{c}} = \frac{1}{2}\epsilon^{\dot{b}\dot{c}\dot{e}}\epsilon^{abc}M_{\dot{e}}^{-1\dot{d}}\left(F_{bc\dot{d}} + g\epsilon_{\dot{d}\dot{f}\dot{g}}B_b{}^{\dot{f}}B_c{}^{\dot{g}} - g\epsilon_{bcd}(\partial_{\dot{d}}X^I)\mathcal{D}^dX^I\right), \tag{3.4}$$

where

$$F_{ab\dot{c}} = \partial_a A_{b\dot{c}} - \partial_b A_{a\dot{c}} + \partial_{\dot{c}}A_{ab}\,, \tag{3.5}$$

and $A_{ab}(x,y)$ was introduced when integrating the total derivative. Eq.(3.4) reduces to the linear self-dual relation (2.37) in the $g \to 0$ limit. One may define a field strength as a scalar quantity with respect to the area-preserving diffeomorphisms made only from the two-form gauge fields which reduces to the linear field strength in the $g \to 0$ limit:[5]

$$\mathcal{H}_{ab\dot{c}} = M_{\dot{c}}{}^{-1\dot{d}}(F_{ab\dot{d}} + g\epsilon_{\dot{d}\dot{e}\dot{f}}B_a{}^{\dot{e}}B_b{}^{\dot{f}}). \tag{3.6}$$

Then, eq.(3.4) takes the following form:

$$\mathcal{H}^{a\dot{b}\dot{c}} = \tilde{\mathcal{H}}^{a\dot{b}\dot{c}} + g\epsilon^{\dot{b}\dot{c}\dot{e}}M_{\dot{e}}{}^{-1\dot{d}}(\partial_{\dot{d}}X^I)\mathcal{D}^aX^I. \tag{3.7}$$

The appropriate gauge transformation law for A_{ab} which achieves the correct transformation property required by eq.(3.4) is given as [1]:

$$\begin{aligned} \delta_\Lambda A_{ab} &= \partial_a\Lambda_b - \partial_b\Lambda_a + g(\kappa^{\dot{c}}\partial_{\dot{c}}A_{ab} + A_{a\dot{c}}\partial_b\kappa^{\dot{c}} - A_{b\dot{c}}\partial_a\kappa^{\dot{c}}) \\ &= \partial_a\Lambda_b - \partial_b\Lambda_a + g(\kappa^{\dot{c}}\partial_{\dot{c}}A_{ab} - (\partial_bA_{a\dot{c}})\kappa^{\dot{c}} + (\partial_aA_{b\dot{c}})\kappa^{\dot{c}}) \\ &\quad +\partial_b(gA_{a\dot{c}}\kappa^{\dot{c}}) - \partial_a(gA_{b\dot{c}}\kappa^{\dot{c}}) \\ &= \partial_a\Lambda_b - \partial_b\Lambda_a + gF_{ab\dot{c}}\kappa^{\dot{c}} + \partial_b(gA_{a\dot{c}}\kappa^{\dot{c}}) - \partial_a(gA_{b\dot{c}}\kappa^{\dot{c}}). \end{aligned} \tag{3.8}$$

The last two terms in eq.(3.8) can be absorbed in the redefinition of the gauge transformation parameter Λ_a. The form in the last line in eq.(3.8) is convenient for finding the Seiberg-Witten

[5] In Ref.[1] the authors imposed linear self-dual relations to define $\mathcal{H}_{ab\dot{c}}$. If one follows this reasoning here, one needs to define $\mathcal{H}_{ab\dot{c}}$ using scalar fields, which seems little bit odd. It would be more natural to define the field strength as in eq.(3.6), and interpret eq.(3.4) as modified self-dual relations. Note that in the ordinary description of M5-brane, the self-dual relations are also extended to the non-linear one (the three-form field strength in the ordinary description can be related to a self-dual three-form, which might be closer to their identification). In the case of $\mathcal{H}_{abc}$ discussed later, imposing the linear self-duality does not uniquely fix the definition, because the self-dual relation includes a quadratic term in $\mathcal{H}_{\dot{a}\dot{b}\dot{c}}$, see eq.(3.17). Anyway, it is not necessary to define the field strength $\mathcal{H}_{ab\dot{c}}$ or $\mathcal{H}_{abc}$ at this moment, and one can regard eq.(3.6) as a shorthand notation for the combination (3.7).

map for A_{ab} which will be discussed in the next section. From the gauge transformation law (3.8) as well as (2.12)–(2.14), one can check that inside the parenthesis of the right hand side of eq.(3.4) transforms as a vector in the dotted directions. Then, the multiplication of the matrix M^{-1} converts this vector into a gauge scalar (invariant under the gauge transformation generated by Λ_a, and scalar under the volume-preserving diffeomorphisms (2.16)), which is the same transformation property with the left hand side of eq.(3.4), i.e. $\mathcal{H}^{a\dot{b}\dot{c}}$.

Next I look at eq.(2.28). Multiplying $M_{\dot{a}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{b}\dot{c}}$ to eq.(2.28), one obtains

$$M_{\dot{a}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{b}\dot{c}}\mathcal{D}_a\mathcal{H}^{a\dot{b}\dot{c}} + 2M_{\dot{a}}{}^{\dot{d}}\mathcal{D}_{\dot{d}}\mathcal{H}_{345} = gM_{\dot{a}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{b}\dot{c}}J_B^{\dot{b}\dot{c}}\,. \tag{3.9}$$

As noted before, I only considered the bosonic fields above. The second term in the left hand side of eq.(3.9) is equal to a total derivative:

$$2M_{\dot{a}}{}^{\dot{d}}\mathcal{D}_{\dot{d}}\mathcal{H}_{345} = \frac{1}{g}\partial_{\dot{a}}\left((\det M)^2 - 1\right), \tag{3.10}$$

where in the right hand side the constant was introduced to ensure a smooth limit for $g \to 0$. Here the identity (2.25) has been used, and notice that

$$\mathcal{H}_{345} = \frac{1}{g}(\det M - 1). \tag{3.11}$$

The first term in eq.(3.9) can be rewritten as

$$\begin{aligned}
& M_{\dot{a}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{b}\dot{c}}\mathcal{D}_a\mathcal{H}^{a\dot{b}\dot{c}} \\
= \ & \epsilon^{abc}M_{\dot{a}}{}^{\dot{d}}\mathcal{D}_a(\tilde{\mathcal{H}}_{bc\dot{d}} - g\epsilon_{bcd}M_{\dot{d}}{}^{-1\dot{e}}(\partial_{\dot{e}}X^I)\mathcal{D}^dX^I) \\
& +2M_{\dot{a}}{}^{\dot{d}}\mathcal{D}_a\left(\mathcal{D}^aX_{\dot{d}} - \frac{1}{2}\epsilon^{abc}(\tilde{\mathcal{H}}_{bc\dot{d}} - g\epsilon_{bcd}M_{\dot{d}}{}^{-1\dot{e}}(\partial_{\dot{e}}X^I)\mathcal{D}^dX^I)\right) \\
= \ & \epsilon^{abc}\mathcal{D}_a\left(F_{bc\dot{a}} + g\epsilon_{\dot{k}\dot{g}\dot{a}}B_b{}^{\dot{k}}B_c{}^{\dot{g}} - g\epsilon_{bcd}(\partial_{\dot{a}}X^I)\mathcal{D}^dX^I\right) - 2g(\mathcal{D}_a\partial_{\dot{a}}X^{\dot{d}})D^aX_{\dot{d}} \\
& +2\mathcal{D}_a\left(M_{\dot{a}}{}^{\dot{d}}\left(\mathcal{D}^aX_{\dot{d}} - \frac{1}{2}\epsilon^{abc}(\tilde{\mathcal{H}}_{bc\dot{d}} - g\epsilon_{bcd}M_{\dot{d}}{}^{-1\dot{e}}(\partial_{\dot{e}}X^I)\mathcal{D}^dX^I)\right)\right).
\end{aligned} \tag{3.12}$$

The last term in eq.(3.12) can be set to zero by the self-dual relation (3.4). Compared with the result in Ref.[1], there is an extra term

$$\begin{aligned}
& -\epsilon^{abc}\mathcal{D}_a(g\epsilon_{bcd}(\partial_{\dot{a}}X^I)\mathcal{D}^dX^I) \\
= \ & -2g\mathcal{D}_a((\partial_{\dot{a}}X^I)\mathcal{D}^aX^I) \\
= \ & -g\partial_{\dot{a}}(D_aX^ID^aX^I) - 2g^2(\partial_{\dot{a}}B_a{}^{\dot{c}})(\partial_{\dot{c}}X^I)D^aX^I - 2g(\partial_{\dot{a}}X^I)D_aD^aX^I.
\end{aligned} \tag{3.13}$$

The second term in the last line in eq.(3.13) cancels a term from the second term in the last but one line in eq.(3.12), the difference from Ref.[1] being the modification in the self-dual relation eq.(3.4). The last term of the last line in eq.(3.13) can be rewritten using the equation of motion

for X^I (2.26), with the fermions being set to zero:

$$\begin{aligned}
& -2g\partial_{\dot a}X^I D_a D^a X^I \\
= \;& 2g(\partial_{\dot a}X^I)\Bigg[\frac{g^4}{2}\{X^{\dot c},X^{\dot d},\{X^{\dot c},X^{\dot d},X^I\}\}+g^4\{X^{\dot c},X^J,\{X^{\dot c},X^J,X^I\}\} \\
& +\frac{g^4}{2}\{X^J,X^K,\{X^J,X^K,X^I\}\}\Bigg].
\end{aligned} \tag{3.14}$$

Now I turn to the right hand side of eq.(3.9). It can be rewritten as

$$\begin{aligned}
& gM_{\dot a}{}^{\dot d}\epsilon_{\dot d\dot b\dot c}J_B^{\dot b\dot c} \\
= \;& g\partial_{\dot a}X^{\dot d}\epsilon_{\dot d\dot b\dot c}\left[g^2(\{X^I,D^{\dot b}X^I,X^{\dot c}\}-(\dot b\leftrightarrow\dot c))-\frac{g^4}{2}\epsilon^{\dot b\dot c\dot e}\{X^I,X^J,\{X^I,X^J,X_{\dot e}\}\}\right] \\
= \;& -2g^5\partial_{\dot a}X^{\dot d}\{X^{\dot c},X^I,\{X^{\dot c},X^I,X^{\dot d}\}\}-g^5\partial_{\dot a}X^{\dot d}\{X^I,X^J,\{X^I,X^J,X^{\dot d}\}\}.
\end{aligned} \tag{3.15}$$

Eq.(3.14) and eq.(3.15) are combined to give

$$\begin{aligned}
& -2g\partial_{\dot a}X^I D_a D^a X^I - gM_{\dot a}{}^{\dot d}\epsilon_{\dot d\dot b\dot c}J_B^{\dot b\dot c} \\
= \;& -g^5\partial_{\dot a}\Bigg[\frac{1}{2}\{X^{\dot c},X^{\dot d},X^I\}\{X^{\dot c},X^{\dot d},X^I\}+\frac{1}{2}\{X^{\dot c},X^I,X^J\}\{X^{\dot c},X^I,X^J\} \\
& +\frac{1}{6}\{X^I,X^J,X^K\}\{X^I,X^J,X^K\}\Bigg] \\
& +2g^5\{X^{\dot c},X^I,(\partial_{\dot a}X^{\dot d})\{X^{\dot c},X^I,X^{\dot d}\}\}+g^5\{X^{\dot c},X^{\dot d},(\partial_{\dot a}X^I)\{X^{\dot c},X^{\dot d},X^I\}\} \\
& +2g^5\{X^{\dot c},X^J,(\partial_{\dot a}X^I)\{X^{\dot c},X^J,X^I\}\}+g^5\{X^I,X^J,(\partial_{\dot a}X^{\dot d})\{X^I,X^J,X^{\dot d}\}\},
\end{aligned} \tag{3.16}$$

where the derivation property of the Nambu-bracket (see eq.(A.6) in the appendix) has been used. The last two lines in eq.(3.17) can also be rewritten as total derivatives similar to the upper lines using eq.(A.7) in the appendix. Thus, I obtain the following non-linearly extended self-dual relation:

$$\begin{aligned}
0 \;=\; & \frac{1}{3}\epsilon^{abc}F_{abc}-g\epsilon^{abc}B_a{}^{\dot b}F_{bc\dot b}-4g^2\det B_a{}^{\dot b}-\frac{g}{2}\mathcal{H}_{a\dot b\dot c}\mathcal{H}^{a\dot b\dot c}+\frac{1}{g}\left((\det M)^2-1\right) \\
& -g\mathcal{D}_aX^I\mathcal{D}^aX^I+g\mathcal{D}_{\dot a}X^I\mathcal{D}^{\dot a}X^I \\
& +\frac{g^5}{2}\{X^{\dot a},X^I,X^J\}\{X^{\dot a},X^I,X^J\}-\frac{g^5}{6}\{X^I,X^J,X^K\}\{X^I,X^J,X^K\}.
\end{aligned} \tag{3.17}$$

Again, if one takes the $g=0$ limit, eq.(3.17) reduces to the linear self-dual relation (2.37). Thus eq.(3.4) and eq.(3.17) are regarded as the non-linear extension of the linear self-dual relations. One may define another component of the field strength with the similar reasonings for the definition (3.6) (see the footnote 3):

$$\mathcal{H}_{abc}=F_{abc}+\frac{g}{2}\epsilon_{abc}\epsilon^{def}B_d{}^{\dot b}F_{ef\dot b}+2g^2\epsilon_{abc}\det B. \tag{3.18}$$

One can check that $\mathcal{H}_{abc}$ in eq.(3.18) transforms as a scalar under the gauge transformations (2.14) and (3.8).

Calculations including the contributions from fermions are similar. The complete results including fermions are included in the appendix A.2.

4 Seiberg-Witten map of BPS conditions for the string solitons

In this section, I examine the Seiberg-Witten map of the BPS conditions for the string solitons on M5-brane which was studied in Ref.[30]. The non-linearly extended self-dual relations (3.4) and (3.17), which include all the bosonic fields in the M5-brane action, are essential since the string solitons involve non-trivial configuration of a scalar field.

Only in this section, the fields in the Nambu-bracket description are denoted with ˆ on them, in order to distinguish them from the corresponding fields in the ordinary description which are denoted without ˆ.

Seiberg-Witten map is a solution to the condition: "Gauge transformations in the Nambu description is compatible with gauge transformations in the ordinary description":

$$\hat{\delta}_{\hat{\Lambda}}\hat{\Phi}(\Phi) = \hat{\Phi}(\Phi + \delta_{\Lambda}\Phi) - \hat{\Phi}(\Phi), \tag{4.1}$$

where $\hat{\Phi}$ (Φ) collectively represents fields in the Nambu-bracket (ordinary) description of M5-brane. The Seiberg-Witten map for the fields $\hat{\varphi}$ ($\hat{\varphi} = \hat{X}^I, \hat{\Psi}$), $\hat{A}^{\dot{a}}$, $\hat{B}_a{}^{\dot{a}}$ and the gauge transformation parameter $\hat{\kappa}^{\dot{a}}$ were obtained in Ref.[6]:

$$\begin{aligned}
\hat{\varphi} &= \varphi + gA^{\dot{a}}\partial_{\dot{a}}\varphi + \mathcal{O}(g^2), \quad (\hat{\varphi} = \hat{X}^I, \hat{\Psi}), && (4.2)\\
\hat{A}^{\dot{a}} &= A^{\dot{a}} + \frac{g}{2}A^{\dot{b}}\partial_{\dot{b}}A^{\dot{a}} + \frac{g}{2}A^{\dot{a}}\partial_{\dot{b}}A^{\dot{b}} + \mathcal{O}(g^2), && (4.3)\\
\hat{B}_a{}^{\dot{a}} &= B_a{}^{\dot{a}} + gA^{\dot{b}}\partial_{\dot{b}}B_a{}^{\dot{a}} - \frac{g}{2}A^{\dot{b}}\partial_a\partial_{\dot{b}}A^{\dot{a}} + \frac{g}{2}A^{\dot{a}}\partial_a\partial_{\dot{b}}A^{\dot{b}} \\
&\quad + g(\partial_{\dot{b}}A^{\dot{b}})B_a{}^{\dot{a}} - g(\partial_{\dot{b}}A^{\dot{a}})B_a{}^{\dot{b}} - \frac{g}{2}(\partial_{\dot{b}}A^{\dot{b}})\partial_a A^{\dot{a}} \\
&\quad + \frac{g}{2}(\partial_{\dot{b}}A^{\dot{a}})\partial_a A^{\dot{b}} + \mathcal{O}(g^2), && (4.4)\\
\hat{\kappa}^{\dot{a}} &= \kappa^{\dot{a}} + \frac{g}{2}A^{\dot{b}}\partial_{\dot{b}}\kappa^{\dot{a}} + \frac{g}{2}(\partial_{\dot{b}}A^{\dot{b}})\kappa^{\dot{a}} - \frac{g}{2}(\partial_{\dot{b}}A^{\dot{a}})\kappa^{\dot{b}} + \mathcal{O}(g^2). && (4.5)
\end{aligned}$$

On the other hand, from the gauge transformation law (3.8) one obtains the Seiberg-Witten map for $\hat{A}_{ab}$:

$$\hat{A}_{ab} = A_{ab} + gA^{\dot{c}}F_{ab\dot{c}} + \mathcal{O}(g^2). \tag{4.6}$$

Here, I have absorbed the last two terms into the redefinition of Λ_a. Notice that the gauge transformations generated by Λ_a do not transform $A_{\dot{a}\dot{b}}$ nor $B_a{}^{\dot{a}}$.

I'd like to examine the BPS conditions for the string solitons which were studied in Ref.[30] (see also Ref.[45]):[6]

$$\mathcal{D}_{\hat{\mu}}\hat{X}^6 + \eta\frac{1}{6}\epsilon_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{\mathcal{H}}_{\hat{\nu}\hat{\rho}\hat{\sigma}} = 0, \tag{4.7}$$

[6] The BPS conditions are crucial to justify our analysis here, with the similar reason explained in Ref.[46] in the case of BPS monopoles on a D3-brane in a constant B-field background. We are planning to present the detail of the scaling arguments to justify the use of the Nambu-bracket M5-brane action, analogous to the zero-slope limit of the open string theory in a constant B-field background discussed in Ref.[20], in our forthcoming paper.

and other fields set to zero, where $\eta = \pm 1$ and $\hat{\mu}, \hat{\nu} = 2, \cdots, 5$. From the Seiberg-Witten map (4.2)–(4.6) as well as the definitions (2.19) and (2.20), one obtains

$$\begin{aligned}
\hat{\mathcal{H}}_{\dot{a}\dot{b}\dot{c}} &= \mathcal{H}_{\dot{a}\dot{b}\dot{c}} + g(A^{\dot{d}}\partial_{\dot{d}}\mathcal{H}_{\dot{a}\dot{b}\dot{c}} + (\partial_{\dot{d}}A^{\dot{d}})\mathcal{H}_{\dot{a}\dot{b}\dot{c}}) + \mathcal{O}(g^2), &(4.8)\\
\hat{\mathcal{H}}_{a\dot{b}\dot{c}} &= \mathcal{H}_{a\dot{b}\dot{c}} + g(A^{\dot{d}}\partial_{\dot{d}}\mathcal{H}_{a\dot{b}\dot{c}} + (\partial_{\dot{d}}A^{\dot{d}})\mathcal{H}_{a\dot{b}\dot{c}}) + \mathcal{O}(g^2), &(4.9)\\
\mathcal{D}_{\dot{a}}\hat{\varphi} &= \partial_{\dot{a}}\varphi + g(A^{\dot{c}}\partial_{\dot{c}}\partial_{\dot{a}}\varphi + (\partial_{\dot{c}}A^{\dot{c}})\partial_{\dot{a}}\varphi) + \mathcal{O}(g^2), &(4.10)\\
\mathcal{D}_{a}\hat{\varphi} &= \partial_{a}\varphi + g(A^{\dot{c}}\partial_{\dot{c}}\partial_{a}\varphi + (\partial_{a}A^{\dot{c}} - B_a{}^{\dot{c}})\partial_{\dot{c}}\varphi) + \mathcal{O}(g^2). &(4.11)
\end{aligned}$$

Using these formulas, from the $\hat{\mu} = 2$ case of eq.(4.7):

$$\mathcal{D}_2\hat{X} + \eta\hat{\mathcal{H}}_{345} = 0, \tag{4.12}$$

where $\hat{X} \equiv \hat{X}^6$, one obtains

$$F_{345} = -\eta(\partial_2 X + \eta g \partial_{\hat{\mu}} X \partial^{\hat{\mu}} X) + \mathcal{O}(g^2), \tag{4.13}$$

where the BPS conditions at $\mathcal{O}(g^0)$ have been used to rewrite the $\mathcal{O}(g)$ term in eq.(4.13). On the other hand, from the $\hat{\mu} = \dot{a}$ case of eq.(4.7):

$$\mathcal{D}_{\dot{a}}\hat{X} - \eta\frac{1}{2}\epsilon_{\dot{a}\dot{b}\dot{c}}\hat{\mathcal{H}}^{2\dot{b}\dot{c}}, \tag{4.14}$$

one obtains

$$F_{2\dot{a}\dot{b}} = \eta\epsilon_{\dot{a}\dot{b}\dot{c}}\partial^{\dot{c}}X + \mathcal{O}(g^2), \tag{4.15}$$

where again the BPS conditions at $\mathcal{O}(g^0)$ have been used to obtain the $\mathcal{O}(g)$ term. Eq.(4.13) and eq.(4.15) should be compared with the results in the ordinary description which were obtained in Ref.[32]:

$$\begin{aligned}
F_{012} + C_{012} &= \eta_1(\sin\theta + \cos\theta\partial_2 X), &(4.16)\\
F_{01\dot{a}} &= \eta_1\cos\theta\partial_{\dot{a}}X, &(4.17)\\
F_{\dot{a}\dot{b}\dot{c}} + C_{\dot{a}\dot{b}\dot{c}} &= \eta_2\epsilon_{\dot{a}\dot{b}\dot{c}}\left(\partial_2 X + \frac{\sin\theta(1 + (\partial_{\hat{\mu}}X\partial^{\hat{\mu}}X))}{\cos\theta - \sin\theta\partial_2 X}\right), &(4.18)\\
F_{2\dot{a}\dot{b}} &= -\eta_2\epsilon_{\dot{a}\dot{b}\dot{c}}\partial^{\dot{c}}X, &(4.19)
\end{aligned}$$

where C_{012} and $C_{\dot{a}\dot{b}\dot{c}}$ are the components of the background C-field in the ordinary description. Eq.(4.13) and eq.(4.15) match with eq.(4.18) and eq.(4.19) respectively in the case $\eta_1\eta_2 = -1$, with the identifications[7]

$$\chi \equiv \eta\theta = g + \mathcal{O}(g^2), \quad \eta = \eta_1. \tag{4.20}$$

To examine whether one can obtain eq.(4.16) and eq.(4.17) from the Nambu-bracket description via the Seiberg-Witten map, one needs to use the non-linearly extended self-dual

[7] This corrects sign errors in Ref.[30]. Notice that the background C-field is specified by $\chi = \eta\theta$, so this identification is independent of the choice of the BPS conditions (the choice of η in eq.(4.7)), as it should be.

relations eq.(3.4) and eq.(3.17) obtained in the previous section. Substituting eq.(4.8)–(4.11) into eq.(3.17), one obtains

$$F_{012} + F_{345} = -g\partial_{\hat{\mu}} X \partial^{\hat{\mu}} X + \mathcal{O}(g^2), \tag{4.21}$$

where the BPS conditions and the self-dual relations at $\mathcal{O}(g^0)$ has been used to rewrite the $\mathcal{O}(g)$ term of eq.(4.21). Again, eq.(4.21) matches with eq.(4.16) and eq.(4.18) with the identifications (4.20), up to $\mathcal{O}(g)$.

To complete the check at $\mathcal{O}(g)$, one should check whether eq.(4.17) can be obtained from the Nambu-bracket description. In order to do this calculation, one needs the expression for the Seiberg-Witten map of $\hat{\mathcal{H}}_{ab\dot{c}}$, which in turn requires the expression for the Seiberg-Witten map of $\hat{A}_{a\dot{b}}$ itself rather than the anti-symmetrized combination of its derivatives $\hat{B}_a{}^{\dot{b}} = \epsilon^{\dot{b}\dot{c}\dot{d}}\partial_{\dot{c}}\hat{A}_{a\dot{d}}$. However, the Seiberg-Witten map of $\hat{A}_{a\dot{b}}$ seems to require a non-local expression [6]. I leave this check to the future work.

5 Discussions

In this paper, the derivation of the non-linearly extended self-dual relations initiated in Ref.[1] was completed by including contributions from all the fields in the Nambu-bracket description of the M5-brane action in a constant C-field background. It is rather impressive that the procedure of Ref.[1] also works with the inclusion of all the fields in the action, though it should work in order for the action to describe M5-brane. This result suggests the existence of a formalism with auxiliary fields and extra local symmetries where the self-dual relations are more manifest, which reduces to the currently discussed action upon gauge fixing. The self-dual relations are characteristic feature of M5-brane, and this result is of essential importance when comparing the Nambu-bracket description of M5-brane to the ordinary description via the Seiberg-Witten map.

To compare the M5-brane action in the ordinary description with the one in the Nambu-bracket description, it will be useful to extend the new auxiliary field formalism also introduced in Ref.[1] to the non-linear DBI-type action in the ordinary description, so that it can be gauge fixed to the form which is more convenient to compare with the M5-brane action in the Nambu-bracket description. For this purpose, it will be useful to understand the introduction the of auxiliary fields in a systematic way. An interesting work in this direction is made in Ref.[47].

In these few years, several new formulations for M-theory branes have been proposed, and it is important to examine to what extent they can describe expected properties of M-theory branes. Consistency with the reduction to type IIA string is a necessary condition [6, 48, 49, 50, 51], but it tends to hide the information of M-theory which we are seeking for. The comparison of the Nambu-bracket description of M5-brane to the ordinary description via the Seiberg-Witten map is a direct check of the former as M-theory brane. The relation to the ordinary description, which can be described in space-time covariant ways, will be important for the Lie 3-algebra to play fundamental role in the description of M-theory. The importance of relating the BLG

model to the covariant formulations was stressed in Ref.[52], see also Ref.[53]. Another trial to relate the BLG model to the light-cone Hamiltonian of M5-brane was made in Ref.[54] without the C-field background, but only the Carrollian limit of the BLG model was obtained. Another approach to M-theory branes is the ABJM model of multiple membranes [55]. In Refs.[56, 57], M5-brane solutions in the ABJM model were constructed. To describe M-theoretical or eleven-dimensional aspects by these M5-brane solutions one tends to encounter non-perturbative problems.[8] Such problems are certainly interesting, but also hard. The approach from the Nambu-bracket description of M5-brane has an advantage that one can see the relation to the ordinary formulation at the classical level through the Seiberg-Witten map. On the other hand, the Seiberg-Witten map has been solved up to the first order in the expansion by the parameter g which characterizes the strength of the interaction through the Nambu-bracket. This is certainly not satisfactory, and one would like to obtain all order expression. Though interacting nature of the M2-brane worldvolume theory makes the analysis complicated compared with the open string worldsheet theory on a D-brane in a constant B-field background, results in that case (see e.g. Refs.[40, 41, 42, 43, 44, 58, 59, 60]) will give clues for how to obtain all order expression of the Seiberg-Witten map in the case of M5-brane in a constant C-field background. Another issue which calls for better understanding is that in the case of a D-brane in a constant B-field background, the product of fields is given by Moyal product, whereas the Nambu-bracket is an analogue of the Poisson-bracket, and the product is not defined. To obtain the Moyal product description of D4-brane from M5-brane via a compactification on a circle, the Nambu-bracket should be deformed appropriately. This issue is discussed in Ref.[61].

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[8] I thank Seiji Terashima for explaining this point.

A Appendix

A.1 Convention for the totally anti-symmetric tensor ϵ^{abc}

Convention (metric $\eta^{ab} = \mathrm{diag}(-++)$):

$$\epsilon^{012} = -\epsilon_{012} = 1. \tag{A.1}$$

It follows that

$$\frac{1}{2}\epsilon_{abc}\epsilon^{dbc} = -\delta_a^d, \quad \epsilon_{abc}\epsilon^{dec} = -\delta_a^d\delta_b^e + \delta_a^e\delta_b^d\,. \tag{A.2}$$

The determinant of a matrix $B_a{}^{\dot{b}}$ can be written as

$$\det B_a{}^{\dot{b}} = \frac{1}{6}\epsilon^{abc}\epsilon_{\dot{a}\dot{b}\dot{c}}B_a{}^{\dot{a}}B_b{}^{\dot{b}}B_c{}^{\dot{c}}\,. \tag{A.3}$$

A.2 Complete form of the self-dual relations

$$\mathcal{H}^{a\dot{b}\dot{c}} = \frac{1}{2}\epsilon^{\dot{b}\dot{c}\dot{e}}\epsilon^{abc}M^{-1\dot{d}}_{\dot{e}}\left(F_{bc\dot{d}} + g\epsilon_{\dot{d}\dot{f}\dot{g}}B_b{}^{\dot{f}}B_c{}^{\dot{g}} - g\epsilon_{bcd}\left((\partial_{\dot{d}}X^I)\mathcal{D}^dX^I + i(\partial_{\dot{d}}\bar{\Psi})\Gamma^d\Psi\right)\right), \tag{A.4}$$

$$\begin{aligned} 0 &= \frac{1}{3}\epsilon^{abc}F_{abc} - g\epsilon^{abc}B_a{}^{\dot{b}}F_{bc\dot{b}} - 4g^2\det B_a{}^{\dot{b}} - \frac{g}{2}\mathcal{H}_{a\dot{b}\dot{c}}\mathcal{H}^{a\dot{b}\dot{c}} + \frac{1}{g}\left((\det M)^2 - 1\right) \\ &\quad -g\mathcal{D}_aX^I\mathcal{D}^aX^I + g\mathcal{D}_{\dot{a}}X^I\mathcal{D}^{\dot{a}}X^I \\ &\quad +\frac{g^5}{2}\{X^{\dot{a}}, X^I, X^J\}\{X^{\dot{a}}, X^I, X^J\} - \frac{g^5}{6}\{X^I, X^J, X^K\}\{X^I, X^J, X^K\} \\ &\quad -ig\bar{\Psi}\Gamma^{\dot{a}}\mathcal{D}_{\dot{a}}\Psi - ig^3\bar{\Psi}\Gamma_{\dot{a}I}\{X^{\dot{a}}, X^I, \Psi\} + \frac{ig^3}{2}\bar{\Psi}\Gamma_{IJ}\Gamma_{345}\{X^I, X^J, \Psi\}. \end{aligned} \tag{A.5}$$

A.3 Some formulas for the Nambu-bracket

Derivation property:

$$\{A, B, CD\} = \{A, B, C\}D + C\{A, B, D\}. \tag{A.6}$$

A frequently used formula:

$$\{A\partial_{\dot{a}}B, C, D\} + \{A\partial_{\dot{a}}C, D, B\} + \{A\partial_{\dot{a}}D, B, C\} = \partial_{\dot{a}}\left(A\{B, C, D\}\right). \tag{A.7}$$

Here, it is assumed that A, B, C, D are bosonic quantities. When some of them are fermionic, one should assign sign factors appropriately. Eq.(A.7) can be obtained from the identity for the totally anti-symmetric tensor $\epsilon^{\dot{a}\dot{b}\dot{c}}$:

$$\delta_{\dot{a}}^{\dot{b}}\epsilon^{\dot{f}\dot{c}\dot{d}} + \delta_{\dot{a}}^{\dot{c}}\epsilon^{\dot{f}\dot{d}\dot{b}} + \delta_{\dot{a}}^{\dot{d}}\epsilon^{\dot{f}\dot{b}\dot{c}} = \delta_{\dot{a}}^{\dot{f}}\epsilon^{\dot{b}\dot{c}\dot{d}}\,. \tag{A.8}$$

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